Lecture 18

Friday, November 15, 2019 5:39 AM

- Do Baby Cauchy IF from Lecture 17 notes.

Thm 1. Let f be analytic in $B(a,R)$. Then, $\exists$ unique power series at a s.t.

$$f(z)=\sum_{n=0}^{\infty} a_n (z-a)^n \quad , \quad z\in B(a,R)$$

w/ radius of conv. $\geq R$. Moreover, f is ∞ $\mathbb{C}$-diff. and

$$a_n = \frac{f^{(n)}(a)}{n!} \qquad (\Rightarrow \text{uniqueness.})$$

Pf. Pick $0 < r < R$. Let $\gamma : [0,2\pi] \to B(a,R)$ be circle $|z-a|=r$ traversed once in + direction ($\gamma(t)=a+re^{it}$, $t\in[0,2\pi]$).

By Baby Cauchy,
$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{\zeta - z} \quad , \quad z\in B(a,r) \quad (1)$$

Fix $z\in B(a,r)$. For $\zeta\in\{\gamma\}$, we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta-a-(z-a)} = \frac{1}{\zeta-a}\,\frac{1}{1-\left(\frac{z-a}{\zeta-a}\right)} = \left\{ \left|\frac{z-a}{\zeta-a}\right| \leq \frac{r}{r_0} < 1 \right\}$$

$$= \frac{1}{\zeta-a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta-a}\right)^n$$ w/ uniform conv. (by Weierstrass) for $\zeta\in\{\gamma\}$.

Thus, we can the exchange $\sum$ and $\int$ in (1) $\Rightarrow$

$$f(z)=\sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}}\right](z-a)^n = \sum_{n=0}^{\infty} a_n (z-a)^n \quad (2)$$

Moreover, $$\left|\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}}\right| \leq \frac{M}{r^n}, \quad (3)$$

Moreover, $\left| \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)\,d\zeta}{(\zeta-a)^{n+1}} \right| \le \frac{M}{r^n}$, (3)

where $M = \sup_{\zeta \in \{\gamma\}} |f(\zeta)| < \infty$. $\Rightarrow$ The R.O.C. in (2) is $\geq r$.

Since $r<R$ is arbitrary we conclude R.O.C. $\geq R$.

Since $f(z)$ is given by power series (2) in $B(a,R)$, $f(z)$ is ∞ $\mathbb{C}$-diff. and, by prior results on power series,

$$a_n = \frac{f^{(n)}(a)}{n!}. \quad \blacksquare$$

<u>Note</u>: If f analytic in G, $\overline{B(a,r)} \subseteq G$, then by Leibniz + Baby Cauchy IF $\Rightarrow$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{|\zeta-a|=r} \frac{f(\zeta)}{(\zeta-z)^{n+1}}\, d\zeta, \quad z \in B(a,r).$$

<u>Cor 1</u>. (Cauchy Estimates) If f analytic and bounded $|f| \le M$ in $B(a,R)$, then

$$|f^{(n)}(a)| \le \frac{n!\, M}{R^n}$$

<u>Prop 1</u>. If f is analytic in $B(a,R)$, then $\exists F$ analytic in $B(a,R)$ s.t. $F' = f$.

<u>Pf</u>. f is given by $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. Consider $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$. This has same R.O.C. $\geq R$ as the power series of f. Thus, F is analytic in $B(a,R)$.

The series can be differentiated term by term inside $B(a,R)$

The series can be differentiated term by term inside $B(a,R)$
$\Rightarrow F' = f$. ☒

Cor 2 (Baby Cauchy Thm). If f is analytic in $B(a,R)$ and $\gamma : [a,b] \to B(a,R)$ is a p-w smooth, closed curve ($\gamma(a)=\gamma(b)$), then $\int_\gamma f\,dz = 0$.